

ON THE RADIUS OF ANALYTICITY OF SOLUTIONS TO THE CUBIC SZEGÖ EQUATION

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ABSTRACT. This paper is concerned with the cubic Szegő equation

$$i\partial_t u = \Pi(|u|^2 u),$$

defined on the L^2 Hardy space on the one-dimensional torus \mathbb{T} , where $\Pi : L^2(\mathbb{T}) \rightarrow L^2_+(\mathbb{T})$ is the Szegő projector onto the non-negative frequencies. For analytic initial data, it is shown that the solution remains spatial analytic for all time $t \in (-\infty, \infty)$. In addition, we find a lower bound for the radius of analyticity of the solution. Our method involves Gevrey estimates based on the ℓ^1 norm of Fourier transforms.

1. INTRODUCTION

In studying the nonlinear Schrödinger equation

$$i\partial_t u + \Delta u = \pm |u|^2 u, \quad (t, x) \in \mathbb{R} \times M, \quad (1.1)$$

Burq, Gérard and Tzvetkov [2] observed that dispersion properties are strongly influenced by the geometry of the underlying manifold M . In [7], Gérard and Grellier mentioned, if there exists a smooth local in time flow map on the Sobolev space $H^s(M)$, then the following Strichartz-type estimate must hold:

$$\|e^{it\Delta} f\|_{L^4([0,1] \times M)} \lesssim \|f\|_{H^{s/2}(M)}. \quad (1.2)$$

It is shown in [1, 2] that, on the two-dimensional sphere, the infimum of the number s such that (1.2) holds is $\frac{1}{4}$; however, if $M = \mathbb{R}^2$, the inequality (1.2) is valid for $s = 0$. As pointed out in [7], this can be interpreted as a lack of dispersion properties for the spherical geometry. Taking this idea further, it is remarked in [7] that dispersion disappears completely when M is a sub-Riemannian manifold (for instance, the Heisenberg group).

As a toy model to study non-dispersive Hamiltonian equation, Gérard and Grellier [7] introduced the *cubic Szegő equation* :

$$i\partial_t u = \Pi(|u|^2 u), \quad (t, \theta) \in \mathbb{R} \times \mathbb{T}, \quad (1.3)$$

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on $L_+^2(\mathbb{T})$, where $\mathbb{T} = [0, 2\pi]$ is the one-dimensional torus, which is identical to the unit circle in the complex plane. Notice that $L_+^2(\mathbb{T})$ is the L^2 Hardy space which is defined by

$$L_+^2(\mathbb{T}) = \left\{ u = \sum_{k \in \mathbb{Z}} u_k e^{ik\theta} \in L^2(\mathbb{T}) : u_k = 0 \text{ for all } k < 0 \right\}. \quad (1.4)$$

Furthermore, in (1.3), the operator $\Pi : L^2(\mathbb{T}) \rightarrow L_+^2(\mathbb{T})$ is the Szegő projector onto the non-negative frequencies, i.e.,

$$\Pi \left(\sum_{k \in \mathbb{Z}} v_k e^{ik\theta} \right) = \sum_{k \geq 0} v_k e^{ik\theta}.$$

Taking the Fourier transform instead of the Fourier series, one can analogously define the Szegő equation on

$$L_+^2(\mathbb{R}) = \{ \phi \in L^2(\mathbb{R}) : \text{supp } \hat{\phi} \subset [0, \infty) \}.$$

In [16], Pocovnicu constructed explicit spatially real analytic solutions for the cubic Szegő equation defined on $L_+^2(\mathbb{R})$. For the initial datum $u_0 = \frac{2}{x+i} - \frac{4}{x+2i}$, it was discovered that one of the poles of the explicit real analytic solution $u(t, x)$ approaches the real line as $|t| \rightarrow \infty$; more precisely, the imaginary part of a pole decreases in the speed $O(\frac{1}{t})$. Thus, the radius of analyticity of $u(t, x)$ shrinks linearly to zero, as $|t| \rightarrow \infty$. This phenomenon gives rise to the following questions: for analytic initial data, does the solution remain spatial analytic for all time? If so, can one estimate, from below, the radius of analyticity? In this manuscript, we attempt to answer these questions by employing the technique of the special Gevrey class of analytic functions.

The Gevrey classes of real analytic functions are characterized by an exponential decay of their Fourier coefficients. In particular, if we set $A := \sqrt{I - \Delta}$, the Gevrey classes are defined by $\mathcal{D}(A^s e^{\sigma A})$, which consists of all L^2 functions u such that $\|A^s e^{\sigma A} u\|_{L^2(\mathbb{T})}$ is finite, where s, σ are non-negative real numbers. If $\sigma = 0$, then $\mathcal{D}(A^s e^{\sigma A}) = \mathcal{D}(A^s) \cong H^s(\mathbb{T})$. However, if $\sigma > 0$, then $\mathcal{D}(A^s e^{\sigma A})$ is the set of real analytic functions with the radius of analyticity bounded below by σ . Also notice, $\mathcal{D}(A^s e^{\sigma A})$ is a Banach algebra provided $s > \frac{1}{2}$ for 1D [5].

The method of Gevrey estimates has been extensively used in literature to establish regularity results for nonlinear evolution equations. It was first introduced for the periodic Navier-Stokes equations in [6], and studied later in the whole space in [14], moreover it was extended to nonlinear analytic parabolic PDE's in [5], and for Euler equations in [10, 12, 11] (see also references therein). Recently, this method was also applied to establish analytic solutions for quasilinear wave equations [9].

In this paper, we employ a special Gevrey class based on the ℓ^1 norm of Fourier transforms. For a given function $u \in L^1(\mathbb{T})$, $u = \sum_{k \in \mathbb{Z}} u_k e^{ik\theta}$, $\theta \in \mathbb{T}$, then the ℓ^1

norm of its Fourier transform is given by

$$\|\hat{u}\|_{\ell^1} = \sum_{k \in \mathbb{Z}} |u_k|. \quad (1.5)$$

Notice that the space ℓ^1 of Fourier transforms is a Banach algebra.

Based on the ℓ^1 norm of Fourier transforms, the following special Gevrey norm is defined in [15]:

$$\|\hat{u}\|_{G_\sigma(\ell^1)} = \sum_{k \in \mathbb{Z}} e^{\sigma|k|} |u_k|, \quad \sigma \geq 0. \quad (1.6)$$

If $u \in L^1(\mathbb{T})$ such that $\|\hat{u}\|_{G_\sigma(\ell^1)} < \infty$, then we write $\hat{u} \in G_\sigma(\ell^1)$.

It is known that the Gevrey class $G_\sigma(\ell^1)$ is a Banach algebra [15], and more importantly, it characterizes the real analytic functions if $\sigma > 0$. In particular, we recall the following result:

Theorem 1.1. [15] *A function $u \in C^\infty(\mathbb{T})$ is real analytic with uniform radius of analyticity ρ , if and only if, $\hat{u} \in G_\sigma(\ell^1)$, for every $0 < \sigma < \rho$.*

In other words, the above theorem says, if $\hat{u} \in G_\sigma(\ell^1)$ with $\sigma > 0$, then u is real analytic and σ is a lower bound for the radius of analyticity of u .

By investigating the steady state of the cubic nonlinear Schrödinger equation, it is demonstrated in [15] that, by employing the Gevrey class $G_\sigma(\ell^1)$, one can obtain a more accurate estimate of the lower bound of the radius of analyticity of solutions to differential equations, compared to the estimate derived from using the regular Gevrey classes $\mathcal{D}(A^s e^{\sigma A})$ (see also the discussion in [9]). Such discovery is verified again in this paper, since we find that, in studying the cubic Szegö equation, the Gevrey class method, based on $G_\sigma(\ell^1)$, provides an estimate of the lower bound of the analyticity radius of the solution, which has a substantially slower shrinking rate, than the estimate obtained from using the Gevrey classes $\mathcal{D}(A^s e^{\sigma A})$. One may refer to Remark 2.5 for this comparison.

Throughout, we study the cubic Szegö equation defined on the torus \mathbb{T} . However, by using Fourier transforms instead of Fourier series, our techniques are also applicable to the same equation defined on the real line, and similar Gevrey regularity results and estimates can be obtained as well (see also ideas from [14]).

To this end, we mention the following existence result, proved in [7].

Theorem 1.2. [7] *Given $u_0 \in H_+^s(\mathbb{T})$, for some $s \geq \frac{1}{2}$, then the cubic Szegö equation (1.3) has a unique solution $u \in C(\mathbb{R}, H_+^s(\mathbb{T}))$.*

2. MAIN RESULT

Before we state the main result, the following proposition should be mentioned.

Proposition 2.1. *Assume $u_0 \in H_+^s(\mathbb{T})$, for some $s > 1$. Let u be the unique global solution of (1.3), furnished by Theorem 1.2. Then,*

$$\|\hat{u}(t)\|_{\ell^1} \leq C(s) \|u_0\|_{H^s}, \quad \text{for all } t \in \mathbb{R}. \quad (2.1)$$

Proof. Notice that, it has been shown in [7] that the cubic Szegő equation admits a Lax pair, and thus the trace norm $Tr(|H_{u(t)}|)$ is a conserved quantity, where $H_u : L_+^2(\mathbb{T}) \rightarrow L_+^2(\mathbb{T})$ is the Hankel operator of u , defined by

$$H_u(h) = \Pi(u\bar{h}). \quad (2.2)$$

Observe that the inequality

$$\|\hat{u}\|_{\ell^1} \leq 3Tr(|H_u|), \quad (2.3)$$

holds for every $u \in L_+^2(\mathbb{T})$, with $\hat{u} \in \ell^1$. For the sake of completion we provide a straightforward proof of (2.3) in the Appendix.

As it has been mentioned in [7], H^s is embedded in the Besov space $B_{1,1}^1$, when $s > 1$, and $Tr(|H_u|)$ is equivalent to the $B_{1,1}^1$ norm of u . Therefore, by (2.3) and the fact that $Tr(|H_u|)$ is conserved, we conclude that the estimate (2.1) holds. \square

Now, we state the main result of this paper.

Theorem 2.2. *Assume $u_0 \in L_+^2(\mathbb{T})$ with $\hat{u}_0 \in G_\sigma(\ell^1)$, for some $\sigma > 0$, then the unique solution $u(t)$ of (1.3), furnished by Theorem 1.2, has the Gevrey regularity $\hat{u}(t) \in G_{\tau(t)}(\ell^1)$, for all $t \in \mathbb{R}$, satisfying*

$$\|\hat{u}(t)\|_{G_{\tau(t)}(\ell^1)} \leq \|\hat{u}_0\|_{G_\sigma(\ell^1)} + 2e^2 \int_0^{|t|} (\|\hat{u}(t')\|_{\ell^1} + 1)^3 dt', \quad (2.4)$$

where $\tau(t) = \sigma e^{-\int_0^{|t|} h(t') dt'}$, and $h(t)$ is defined in (2.19), below. In particular, one has $\tau(t) \geq \text{const} \cdot e^{-t^2}$.

Remark 2.3. Essentially, Theorem 2.2 shows the persistency of the spatial analyticity of the solution $u(t)$ for all time $t \in (-\infty, \infty)$ provided the initial datum is analytic. Also, it implies that the radius of spatial analyticity of $u(t)$ might shrink at most exponentially fast as $|t| \rightarrow \infty$.

Proof. Due to the assumption on the initial datum u_0 , we know that u_0 is real analytic by Theorem 1.1, and hence $u_0 \in H_+^p(\mathbb{T})$, for every non-negative real number p , in particular for $p \geq \frac{1}{2}$. Therefore, the global existence and uniqueness of the solution $u \in C(\mathbb{R}, H_+^s(\mathbb{T}))$ are guaranteed by Theorem 1.2, for $s \geq \frac{1}{2}$.

Throughout, we focus on the positive time $t \geq 0$. By replacing t by $-t$, the same proof works for the negative time.

We shall implement the Galerkin approximation method. Recall the cubic Szegő equation is defined on the Hardy space $L_+^2(\mathbb{T})$ with a natural basis $\{e^{ik\theta}\}_{k \geq 0}$. Denote

by P_N the projection onto the span of $\{e^{ik\theta}\}_{0 \leq k \leq N}$. We let

$$u_N(t) = \sum_{k=0}^N u_{N,k}(t) e^{ik\theta} \quad (2.5)$$

be the solution of the Galerkin system:

$$i\partial_t u_N = P_N (\Pi(|u_N|^2 u_N)) \quad (2.6)$$

with the initial condition $u_N(0) = P_N u_0$. We see that (2.6) is an N -dimensional system of ODE with a cubic non-linearity, and thus it has a unique solution on some time interval $[0, T_N]$, for some $T_N > 0$, with $u_{N,k} \in C^1[0, T_N]$.

Since u_N is a solution of (2.6), and thanks to the fact that H^s is an algebra, when $s > \frac{1}{2}$ for 1D, straightforward calculations show that

$$\frac{d}{dt} \|u_N(t)\|_{H^s} \leq C \|u_N(t)\|_{H^s}^3, \quad t \in [0, T_N]. \quad (2.7)$$

Integrating (2.7) on $[0, t]$ yields

$$\|u_N(t)\|_{H^s} \leq \|u_0\|_{H^s} + C \int_0^t \|u_N(t')\|_{H^s}^3 dt', \quad (2.8)$$

where we have used $\|u_N(0)\|_{H^s} = \|P_N u_0\|_{H^s} \leq \|u_0\|_{H^s}$. Hence, by using a standard comparison theorem, (2.8) shows that $\|u_N(t)\|_{H^s} \leq y(t)$, where $y(t) = (\|u_0\|_{H^s}^{-2} - 2Ct)^{-\frac{1}{2}}$, is the solution of the Volterra integral equation $y(t) = \|u_0\|_{H^s} + C \int_0^t y(t')^3 dt'$. Although $y(t)$ blows up in finite time, nonetheless, there exists a time $0 < T^* < T_N$ such that

$$\|u_N(t)\|_{H^s} \leq y(t) \leq C_1 \quad \text{for all } t \in [0, T^*], \quad (2.9)$$

where C_1 is independent of N . Moreover, since u_N satisfies (2.6), it follows that

$$\|\partial_t u_N(t)\|_{H^s} \leq \|u_N(t)\|_{H^s}^3 \leq C_1^3, \quad \text{for all } t \in [0, T^*], \quad (2.10)$$

provided $s > \frac{1}{2}$. By virtue of (2.9), (2.10) and thanks to Aubin-Lions-Simon compactness theorem [18], we have, up to a subsequence, $u_N \rightarrow \tilde{u}$ weak-* in $L^\infty(0, T^*; H^s)$, and strongly in $C([0, T^*]; H^{s-\epsilon})$, for every $\epsilon > 0$. Then, it is straightforward to check, by letting $N \rightarrow \infty$, that \tilde{u} is a weak solution of the cubic Szegő equation (1.3) on $[0, T^*]$. Since u is the unique global solution furnished by Theorem 1.2, one must have $u = \tilde{u}$ on $[0, T^*]$.

Now, let us fix an arbitrary $T > 0$. Since u is a global unique solution, we can iterate the above argument and obtain that the sequence $u_N(t)$, which solves (2.6) on $[0, T]$, satisfies the property that the $L^\infty([0, T])$ norms of the sequences $\|u_N(t)\|_{H^s}$ and $\|\partial_t u_N(t)\|_{H^s}$ are both uniformly bounded, with respect to N . Recall that the space ℓ^1 is endowed with the norm (1.5), and thus ℓ^1 is compactly embedded in H^s ,

for every $s > \frac{1}{2}$. Therefore, due to Aubin-Lions-Simon compactness theorem [18], up to a subsequence, one also has

$$\hat{u}_N \rightarrow \hat{u}, \quad \text{strongly in } C([0, T]; \ell^1). \quad (2.11)$$

Due to (2.5) and (2.6), we infer

$$\frac{d}{dt} u_{N,k}(t) = -i \sum_{\substack{n-j+m=k \\ 0 \leq n, j, m \leq N}} u_{N,n}(t) \bar{u}_{N,j}(t) u_{N,m}(t), \quad t \in [0, T], \quad k = 0, 1, \dots, N. \quad (2.12)$$

By (2.12), one can easily find that

$$\frac{d}{dt} |u_{N,k}(t)| \leq \sum_{\substack{n-j+m=k \\ 0 \leq n, j, m \leq N}} |u_{N,n}(t)| |u_{N,j}(t)| |u_{N,m}(t)|, \quad t \in [0, T], \quad k = 0, 1, \dots, N. \quad (2.13)$$

Let $\tau(t) \in C^1[0, T]$, to be determined later. In order to estimate the Gevrey norm, defined in (1.6), we consider

$$\begin{aligned} & \frac{d}{dt} (e^{\tau(t)k} |u_{N,k}(t)|) \\ &= \tau'(t) k e^{\tau(t)k} |u_{N,k}(t)| + e^{\tau(t)k} \frac{d}{dt} |u_{N,k}(t)| \\ &\leq \tau'(t) k e^{\tau(t)k} |u_{N,k}(t)| + e^{\tau(t)k} \sum_{\substack{n-j+m=k \\ 0 \leq n, j, m \leq N}} |u_{N,n}(t)| |u_{N,j}(t)| |u_{N,m}(t)|, \end{aligned}$$

for $k = 0, 1, \dots, N$, and $t \in [0, T]$, where (2.13) has been used in the last inequality.

Now, summing over all integers $k = 0, 1, \dots, N$ yields

$$\begin{aligned} & \frac{d}{dt} \left(\sum_{k=0}^N e^{\tau(t)k} |u_{N,k}(t)| \right) \\ &\leq \tau' \sum_{k=0}^N k e^{\tau k} |u_{N,k}| + \sum_{k=0}^N e^{\tau k} \left(\sum_{\substack{n-j+m=k \\ 0 \leq n, j, m \leq N}} |u_{N,n}| |u_{N,j}| |u_{N,m}| \right) \\ &= \tau' \sum_{k=0}^N k e^{\tau k} |u_{N,k}| + \sum_{k=0}^N \left(\sum_{\substack{n-j+m=k \\ 0 \leq n, j, m \leq N}} e^{\tau n} |u_{N,n}| e^{-\tau j} |u_{N,j}| e^{\tau m} |u_{N,m}| \right) \\ &\leq \tau' \sum_{k=0}^N k e^{\tau k} |u_{N,k}| + \left(\sum_{k=0}^N e^{\tau k} |u_{N,k}| \right)^2 \left(\sum_{k=0}^N |u_{N,k}| \right), \end{aligned} \quad (2.14)$$

where the last formula is obtained by using the Young's convolution inequality and the fact $e^{-\tau j} \leq 1$, for $\tau, j \geq 0$.

Now, we estimate the second term on the right-hand side of (2.14). The key ingredient of the calculation is the elementary inequality $e^x \leq e + x^\ell e^x$, for all $x \geq 0$, $\ell \geq 0$, and we select $\ell = \frac{1}{2}$ here. Hence

$$\begin{aligned} & \left(\sum_{k=0}^N e^{\tau k} |u_{N,k}| \right)^2 \left(\sum_{k=0}^N |u_{N,k}| \right) \\ & \leq \left(\sum_{k=0}^N e |u_{N,k}| + \sum_{k=0}^N \tau^{\frac{1}{2}} k^{\frac{1}{2}} e^{\tau k} |u_{N,k}| \right)^2 \left(\sum_{k=0}^N |u_{N,k}| \right) \\ & \leq 2e^2 \left(\sum_{k=0}^N |u_{N,k}| \right)^3 + 2\tau \left(\sum_{k=0}^N k e^{\tau k} |u_{N,k}| \right) \left(\sum_{k=0}^N e^{\tau k} |u_{N,k}| \right) \left(\sum_{k=0}^N |u_{N,k}| \right), \end{aligned} \quad (2.15)$$

where we have used Young's inequality and Hölder's inequality.

By recalling the Gevrey norm $\|\hat{u}_N(t)\|_{G_{\tau(t)}(\ell^1)} = \sum_{k=0}^N e^{\tau(t)k} |u_{N,k}(t)|$ and the ℓ^1 norm $\|\hat{u}_N(t)\|_{\ell^1} = \sum_{k=0}^N |u_{N,k}(t)|$, we infer from (2.14) and (2.15) that

$$\begin{aligned} & \frac{d}{dt} \|\hat{u}_N(t)\|_{G_{\tau(t)}(\ell^1)} \leq 2e^2 \|\hat{u}_N(t)\|_{\ell^1}^3 \\ & + \left[\tau'(t) + 2\tau(t) \|\hat{u}_N(t)\|_{G_{\tau(t)}(\ell^1)} \|\hat{u}_N(t)\|_{\ell^1} \right] \left(\sum_{k=0}^N k e^{\tau(t)k} |u_{N,k}(t)| \right), \end{aligned} \quad (2.16)$$

for all $t \in [0, T]$. Note that (2.11) implies that there exists $N' \in \mathbb{N}$ such that

$$\|\hat{u}_N(t)\|_{\ell^1} \leq \|\hat{u}(t)\|_{\ell^1} + 1/2, \quad \text{for all } N > N', \quad t \in [0, T]. \quad (2.17)$$

A combination of (2.16) and (2.17) gives

$$\begin{aligned} & \frac{d}{dt} \|\hat{u}_N(t)\|_{G_{\tau(t)}(\ell^1)} \leq 2e^2 (\|\hat{u}(t)\|_{\ell^1} + 1/2)^3 \\ & + \left[\tau'(t) + 2\tau(t) \|\hat{u}_N(t)\|_{G_{\tau(t)}(\ell^1)} (\|\hat{u}(t)\|_{\ell^1} + 1/2) \right] \left(\sum_{k=0}^N k e^{\tau(t)k} |u_{N,k}(t)| \right), \end{aligned} \quad (2.18)$$

for all $N > N'$, and $t \in [0, T]$.

By investigating estimate (2.18), we intend to find a uniform bound for $\|\hat{u}_N(t)\|_{G_{\tau(t)}(\ell^1)}$, on $[0, T]$, with an appropriate formula for $\tau(t)$. Indeed, we set

$$h(t) := 2 \left(\|\hat{u}_0\|_{G_\sigma(\ell^1)} + 2e^2 \int_0^{|t|} (\|\hat{u}(t')\|_{\ell^1} + 1)^3 dt' \right) (\|\hat{u}(t)\|_{\ell^1} + 1), \quad (2.19)$$

and let $\tau(t)$ solves the following differential equation

$$\tau'(t) + \tau(t)h(t) = 0, \quad \tau(0) = \sigma, \quad (2.20)$$

i.e.,

$$\tau(t) = \sigma e^{-\int_0^{|t|} h(t') dt'}. \quad (2.21)$$

Next, we claim that

$$2 \|\hat{u}_N(t)\|_{G_{\tau(t)}(\ell^1)} (\|\hat{u}_N(t)\|_{\ell^1} + 1/2) < h(t), \quad (2.22)$$

for all $t \in [0, T)$, and $N > N'$. Clearly, (2.22) holds at $t = 0$. By continuity, one has (2.22) is valid for a short time $[0, t^*)$, where t^* is the supremum of all t satisfying (2.22). Suppose $t^* < T$ and we seek a contradiction. Combining (2.18), (2.20) and (2.22) yields

$$\frac{d}{dt} \|\hat{u}_N(t)\|_{G_{\tau(t)}(\ell^1)} \leq 2e^2 (\|\hat{u}(t)\|_{\ell^1} + 1/2)^3 \quad (2.23)$$

for all $t \in [0, t^*)$ and $N > N'$. Integrating (2.23) on $[0, t^*]$ shows

$$\|\hat{u}_N(t^*)\|_{G_{\tau(t^*)}(\ell^1)} < \|\hat{u}_0\|_{G_\sigma(\ell^1)} + 2e^2 \int_0^{t^*} (\|\hat{u}(t)\|_{\ell^1} + 1)^3 dt,$$

and along with (2.17), we infer (2.22) holds at $t = t^*$ for $N > N'$, and can thus be extended, in time, beyond t^* , contradicting our definition of t^* . It follows that $t^* = T$, i.e., (2.22) holds on $[0, T)$. Thus (2.23) is also valid on $[0, T)$, and after integrating, we get

$$\|\hat{u}_N(t)\|_{G_{\tau(t)}(\ell^1)} \leq \|\hat{u}_0\|_{G_\sigma(\ell^1)} + 2e^2 \int_0^t (\|\hat{u}(t')\|_{\ell^1} + 1)^3 dt', \quad (2.24)$$

for all $t \in [0, T]$ and $N > N'$.

Next, we aim to show that $\|\hat{u}(t)\|_{G_{\tau(t)}(\ell^1)}$ is also bounded by the right-hand side of (2.24). In fact, since the right-hand side of (2.24) is independent of N , and thanks to Proposition 2.1, one has that the sequence $\left\| \widehat{e^{\tau(\cdot)\mathcal{A}} u_N}(t) \right\|_{\ell^1}$, where $\mathcal{A} = \sqrt{-\partial_{\theta\theta}}$, is a bounded in $L^\infty([0, T])$. Following the tradition in Functional Analysis we denote by c_0 the space of all sequences $\{a_k\}_{k \in \mathbb{Z}}$ of complex numbers such that a_k converges to zero. It is a well-known fact that the space ℓ^1 is the dual of c_0 ([3], page 76). Consequently, $L^\infty([0, T]; \ell^1)$ is the dual of $L^1([0, T]; c_0)$. Therefore, by Alaoglu's theorem, there exist $\hat{v} \in L^\infty([0, T]; \ell^1)$ such that $\{\widehat{e^{\tau(t)\mathcal{A}} u_N}(t)\}$ contains a subsequence that converges to \hat{v} , with respect to the weak-* topology of $L^\infty([0, T]; \ell^1)$. It follows that

$$\begin{aligned} \|\hat{v}\|_{L^\infty([0, T]; \ell^1)} &\leq \liminf_{N \rightarrow \infty} \left\| \widehat{e^{\tau(\cdot)\mathcal{A}} u_N}(\cdot) \right\|_{L^\infty([0, T]; \ell^1)} \\ &= \liminf_{N \rightarrow \infty} \left(\sup_{0 \leq t \leq T} \|\hat{u}_N(t)\|_{G_{\tau(t)}(\ell^1)} \right). \end{aligned} \quad (2.25)$$

On the other hand, by (2.11), $\hat{u}_N \rightarrow \hat{u}$ strongly in $C([0, T]; \ell^1)$, and thus $v(t) = e^{\tau(t)\mathcal{A}}u(t)$. Therefore, we infer from (2.24) and (2.25) that

$$\begin{aligned} \sup_{0 \leq t \leq T} \|\hat{u}(t)\|_{G_{\tau(t)}(\ell^1)} &= \sup_{0 \leq t \leq T} \left\| e^{\widehat{\tau(t)\mathcal{A}}} u(t) \right\|_{\ell^1} = \sup_{0 \leq t \leq T} \|\hat{v}(t)\|_{\ell^1} \\ &\leq \|\hat{u}_0\|_{G_\sigma(\ell^1)} + 2e^2 \int_0^T (\|\hat{u}(t')\|_{\ell^1} + 1)^3 dt'. \end{aligned}$$

In particular, from the above one has

$$\|\hat{u}(T)\|_{G_{\tau(T)}(\ell^1)} \leq \|\hat{u}_0\|_{G_\sigma(\ell^1)} + 2e^2 \int_0^T (\|\hat{u}(t')\|_{\ell^1} + 1)^3 dt'.$$

Therefore, since $T > 0$ is arbitrarily selected, the above implies (2.4) by choosing $T = t$.

Since u_0 is real analytic, the assumption in Proposition 2.1 is satisfied, thus estimate (2.1) holds. Therefore, by (2.1), (2.4), (2.19) and (2.21), we conclude that for large values of $|t|$:

$$\|\hat{u}(t)\|_{G_{\tau(t)}(\ell^1)} \leq \text{const} \cdot |t|, \quad h(t) \leq \text{const} \cdot |t|, \quad \text{and} \quad \tau(t) \geq \text{const} \cdot e^{-t^2}. \quad (2.26)$$

□

Remark 2.4. For analytic initial data, estimate (2.4) of the Gevrey norm $\|\hat{u}(t)\|_{G_{\tau(t)}(\ell^1)}$ can provide a growth estimate of the H^s norm of the solution $u(t)$. Indeed,

$$\|u\|_{H^s}^2 = \sum_{k \geq 0} (k^{2s} + 1) |u_k|^2 \leq \sup |u_k| \left(\sum_{k \geq 0} |u_k| e^{\tau k} \frac{k^{2s}}{e^{\tau k}} + \sum_{k \geq 0} |u_k| \right).$$

Since the maximum of the function $k \mapsto \frac{k^{2s}}{e^{\tau k}}$ occurs at $k = \frac{2s}{\tau}$, we obtain

$$\|u\|_{H^s}^2 \leq \|\hat{u}\|_{\ell^1} \left[e^{-2s} \left(\frac{2s}{\tau} \right)^{2s} \|\hat{u}\|_{G_{\tau}(\ell^1)} + \|\hat{u}\|_{\ell^1} \right]. \quad (2.27)$$

By (2.1), (2.26) and (2.27), it follows that for large values of $|t|$ one has:

$$\|u(t)\|_{H^s}^2 \leq C(s) e^{2st^2} |t|,$$

that is to say, the H^s norm grows exponentially, if $s > \frac{1}{2}$, which agrees with the H^s norm estimates in [7].

Remark 2.5. Let us set $A = \sqrt{I - \Delta}$. Recall the regular Gevrey classes are defined by $\mathcal{D}(A^s e^{\sigma A})$ furnished the norm $\|A^s e^{\sigma A} \cdot\|_{L^2(\mathbb{T})}$, where s and σ are non-negative real numbers. It has been mentioned in the Introduction that we choose to employ the special Gevrey class $G_\sigma(\ell^1)$ in this manuscript, since it provides better estimate of the lower bound the radius of analyticity of the solution. In particular, we can do the following comparisons.

Suppose the initial condition $u_0 \in \mathcal{D}(A^s e^{\sigma A})$, $s > \frac{1}{2}$, $\sigma > 0$, and let us perform the estimates by using the regular Gevrey classes as in [9, 11]. Adopting similar arguments as in Theorem 2.2, one can manage to show that

$$\|A^s e^{\tilde{\tau}(t)A} u(t)\|_{L^2}^2 \leq \|A^s e^{\sigma A} u_0\|_{L^2}^2 + C \int_0^{|t|} (\|u(t')\|_{H^s} + 1)^4 dt', \quad s > \frac{1}{2}$$

if $\tilde{\tau}(t) = \sigma e^{-\int_0^{|t|} \tilde{h}(t') dt'}$, where $\tilde{h}(t) = C_1 \left(\|A^p e^{\sigma A} u_0\|_{L^2}^2 + \int_0^{|t|} (\|u(t')\|_{H^s} + 1)^4 dt' \right)$. Since $\|u(t)\|_{H^s}$, $s > \frac{1}{2}$, has an upper bound that grows exponentially as $|t| \rightarrow \infty$ (see [7]), we infer that $\tilde{\tau}(t)$ might shrink *double* exponentially, compared to the exponential shrinking rate of $\tau(t)$ established in Theorem 2.2, where the Gevrey class $G_\sigma(\ell^1)$ is used. Such advantage of employing the special Gevrey class $G_\sigma(\ell^1)$ stems from the uniform boundedness of the norm $\|\hat{u}(t)\|_{\ell^1}$ for the solution u to the cubic Szegő equation for sufficiently regular initial data.

3. APPENDIX

For the convenience, we give a straightforward proof of the following property of the Hankel operator.

Proposition 3.1. *For any $u \in L_+^2(\mathbb{T})$, such that $\hat{u} \in \ell^1$, the following inequality holds*

$$\|\hat{u}\|_{\ell^1} \leq 3Tr(|H_u|). \quad (3.1)$$

Proof. Recall the following result in the operator theory (see, e.g., [4]). Let A be an operator on a Hilbert space H , where A belongs to the trace class. If $\{e_k\}$ and $\{f_k\}$ are two orthonormal bases for H , then

$$\sum_k |(Ae_k, f_k)| \leq Tr(|A|) := \sum_k (|A|e_k, e_k). \quad (3.2)$$

So, in order to find a lower bound of $Tr(|H_u|)$, we use the fact (3.2) by computing $\sum_k |(H_u e^{ik\theta}, f_k)|$ with the basis $\{f_k\}$ selected below. Note, by definition (2.2) of the Hankel operator $H_u : L_+^2(\mathbb{T}) \rightarrow L_+^2(\mathbb{T})$, we have

$$H_u e^{ik\theta} = \Pi(u e^{-ik\theta}) = \Pi \left(\sum_{j \geq 0} u_j e^{i(j-k)\theta} \right) = \sum_{j \geq k} u_j e^{i(j-k)\theta} = \sum_{j \geq 0} u_{j+k} e^{ij\theta}. \quad (3.3)$$

If we choose $f_k = e^{ik\theta}$, $k \geq 0$, and use (3.3), then it follows that

$$\sum_{k \geq 0} |(H_u e^{ik\theta}, e^{ik\theta})| = \sum_{k \geq 0} \left| \left(\sum_{j \geq 0} u_{j+k} e^{ij\theta}, e^{ik\theta} \right) \right| = \sum_{k \geq 0} |u_{2k}|. \quad (3.4)$$

However, if we select $f_k = e^{i(k+1)\theta}$, for even integer $k \geq 0$; and $f_k = e^{i(k-1)\theta}$ for odd integer $k \geq 1$, then

$$\sum_{k \geq 0} |(H_u e^{ik\theta}, f_k)| = \sum_{k \geq 0 \text{ even}} |u_{2k+1}| + \sum_{k \geq 1 \text{ odd}} |u_{2k-1}| = 2 \sum_{k \geq 0 \text{ even}} |u_{2k+1}|. \quad (3.5)$$

Furthermore, let us choose $f_0 = 1$; $f_k = e^{i(k-1)\theta}$ if $k \geq 2$ is even; and $f_k = e^{i(k+1)\theta}$ if k is odd, then a straightforward calculation yields

$$\sum_{k \geq 0} |(H_u e^{ik\theta}, f_k)| = 2 \sum_{k \geq 1 \text{ odd}} |u_{2k+1}|. \quad (3.6)$$

Finally, we conclude from (3.4)-(3.6) and the fact (3.2) that

$$3Tr(|H_u|) \geq \sum_{k \geq 0} |u_k| = \|\hat{u}\|_{\ell^1}.$$

□

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